Lecture 2
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Steven Anlage

The heart and soul of superconductivity is the Meissner Effect. This feature uniquely distinguishes superconductivity from many other states of matter. Here we discuss some simple phenomenological approaches to describing the Meissner effect quantitatively.

The London equations

The brothers F. and H. London wrote down two simple equations which conveniently incorporate the electrodynamic response of a superconductor. These equations describe the microscopic electric (**E**) and magnetic (**h**) fields inside a superconductor. Here **h** is the microscopic flux density, and B will be the macroscopic averaged flux density. See Appendix 2 (p. 435 of Tinkham).

To derive the first London equation, think of the net force acting on the charge carrier in a normal metal (Drude model):

$$\frac{d(m\mathbf{v})}{dt} = e\mathbf{E} - \frac{m\mathbf{v}}{\tau}$$

(Note that this is a LOCAL equation. It assumes that only the local electric field influences the drift velocity. As such, it requires the mean free path be less than the magnetic penetration depth, $\ell_{MFP} < \lambda_L$). Here **v** is the average or "drift" velocity of the charge carrier of charge *e*, *m* is its mass, **E** is the local electric field, and τ is a phenomenological scattering time for the carrier which describes how long it takes the scattering to bring the velocity of the carrier to zero. In a normal metal in steady state, the drift velocity achieves a constant value, meaning that the electric force and scattering forces balance, leading to:

$$\langle \mathbf{v} \rangle = e\mathbf{E}\,\tau / m$$

If there are *n* carriers per unit volume, the current density can be written as $J = ne\langle \mathbf{v} \rangle$, so

$$\mathbf{J} = \frac{ne^2\tau}{m}\mathbf{E}$$

which is Ohm's Law ($\mathbf{J} = \sigma\mathbf{E}$) with the conductivity $\sigma = ne^2\tau/m$. Ohm's law says that an electric current in a normal metal is a consequence of an applied electric field.

To model a superconductor, we shall suppose that there is a density of superconducting electrons, n_s, and they do not have their velocities reduced to zero by means of scattering. (See Tinkham p. 5 for why $\tau\rightarrow\infty$ does not give perfect conductivity) From the above equation, this means that the electrons will accelerate in an applied electric field! $m\,\partial\vec{v}/\partial t = e\vec{E}$, giving rise to the **first London equation**:

$$\frac{\partial \mathbf{J_s}}{\partial t} = \frac{n_s e^2}{m}\mathbf{E} \qquad \text{or} \qquad \frac{\partial(\Lambda \mathbf{J_s})}{\partial t} = \mathbf{E}$$

Strictly speaking, this equation only holds for ac currents and electric fields, since it predicts very large currents for large times at dc. The first London equation says that in order to create an alternating current (i.e. a non-zero $\partial\vec{J}_s/\partial t$) it is necessary to establish an electric field in the superconductor. This has implications for the finite-frequency losses in superconductors. If any un-paired electrons (quasiparticles) are around, they will be accelerated by the electric field and cause Ohmic dissipation. Hence a superconductor has a small but finite dissipation when illuminated with a finite frequency electromagnetic wave at temperatures above zero Kelvin. Superconductors are only dissipation-less at zero frequency, or at finite frequency at zero temperature (for a fully-gapped superconductor).

We define a new quantity, Λ as,

$$\frac{\partial \mathbf{J_s}}{\partial t} = \frac{1}{\Lambda}\mathbf{E} = \frac{1}{\mu_0 \lambda_L{}^2}\mathbf{E}$$

where $\Lambda = \mu_0\,\lambda_L{}^2 = m/(n_s\,e^2)$. We have also introduced an important new length scale, the (London) magnetic penetration depth, λ_L. It is defined as,

$$\lambda_L = \sqrt{\frac{m}{\mu_0 n_s e^2}}$$

To get a deeper insight into the first London Equation and this new length scale, start with the Maxwell equation for the microscopic fields (Ampere's Law),

$$\nabla \times \vec{B} = \mu_0 \mathbf{J}_s + \mu_0 \frac{\partial \vec{D}}{\partial t}$$

and ignoring the displacement current (this is usually appropriate in superconductors because we often consider only frequencies ω < 2Δ ~ THz roughly), take the time derivative of both sides and use the first London equation, to obtain,

$$\nabla \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \frac{\partial \mathbf{J}_s}{\partial t} = \frac{1}{\lambda_L^2} \mathbf{E}$$

Now take the curl of both sides,

$$\nabla \times \nabla \times \frac{\partial \vec{B}}{\partial t} = \frac{1}{\lambda_L^2} \nabla \times \mathbf{E}$$

The electric field curls around the time-varying magnetic field (Faraday's law)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

to get

$$\nabla \times \nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{1}{\lambda_L^2} \frac{\partial \vec{B}}{\partial t}$$

Integrating both sides with respect to time yields

$$\nabla \times \nabla \times \vec{B} + \frac{1}{\lambda_L^2} \vec{B} = 0.$$

Now use the vector identity

$$\nabla \times \nabla \times \vec{B} = \nabla\left(\nabla \bullet \vec{B}\right) - \nabla^2 \vec{B}$$

And the fact that $\nabla \bullet \vec{B} = 0$ to arrive at

$$\nabla^2 \vec{B} = \frac{1}{\lambda_L^2} \vec{B}$$

This equation admits solutions of the general form $B(x) = B_0 e^{\pm x/\lambda_L}$, so the London penetration depth λ_L represents the exponential screening length of the magnetic field in the superconductor. This length scale is also commonly referred to as the "magnetic penetration depth" for obvious reasons. This equation shows that the magnetic field is excluded from the bulk of a superconductor, and describes the result of the Meissner effect.

A similar result can be derived for the electric field:

$$\nabla^2 \vec{E} = \frac{1}{\lambda_L^2} \vec{E} \,,$$

showing that it is screened on the same length scale. The London penetration depth can be estimated for Al, which has a total carrier density of $n = 18.1 \times 10^{22}$ free electrons/cm^3, where we find $\lambda_L = 12.5$ nm. This is an important microscopic length scale in superconductors.

The Second London equation

Start with London's first equation, $\frac{\partial \mathbf{J_s}}{\partial t} = \frac{n_s e^2}{m} \mathbf{E}$ and take the curl of both sides:

$\frac{\partial}{\partial t} \vec{\nabla} \times \vec{J}_s = \frac{n_s e^2}{m} \vec{\nabla} \times \vec{E}$ and use Faraday's law ($\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$) to write

$\frac{\partial}{\partial t} \left[\vec{\nabla} \times \vec{J}_s + \frac{n_s e^2}{m} \vec{B} \right] = 0$. This is a combination of Faraday's law and Lenz's law. It says that a conductor will develop a current to oppose a change in flux in the material. This equation does not predict the Meissner effect of course, only diamagnetic response to time-varying fields. The London's thought that superconductors could be described by simply making the square-bracketed term equal to zero:

$\vec{\nabla} \times \vec{J}_s = -\frac{n_s e^2}{m} \vec{B}$ (the London hypothesis)

Or, $\Lambda \vec{\nabla} \times \vec{J}_s = -\vec{B}$ **(the second London equation).**

This is not a rigorous derivation of the second London equation. deGennes, for example, uses an energy minimization argument to arrive at the same result.

This equation states that a magnetic field applied to a superconductor creates a screening current, such that the curl of that current is oppositely directed to the field. It says that dc currents are controlled by magnetic fields, as opposed to normal metals where they are controlled by electric fields, $\vec{J}_n = \sigma \vec{E}$.

This equation only applies under limited conditions:

1) B must be "small" and treated as a perturbation, namely $B \ll B_c$.

2) The superfluid density should be uniform in space.

3) Local electrodynamics, namely $\lambda_L \gg \frac{1}{1/\xi_0 + 1/\ell_{mfp}}$.

One can derive both London equations by starting with the following simple expression relating the vector potential to the super-current response, $\vec{J}_s = -\frac{1}{\Lambda}\vec{A}$. Taking the time-derivative of both sides yields London's first equation, while taking the curl of both sides yields the second London equation. However one must choose an appropriate gauge for the vector potential, and the standard convention is the London gauge: a) $\Lambda\vec{J}_s = -\vec{A}$ is obeyed on all surfaces and in the bulk of the superconductor, b) $\nabla \bullet \vec{A} = \nabla \bullet \vec{J}_s = 0$, c) $\vec{A} \to 0$ deep (many penetration depths) inside the superconductor. These conditions can break down in a number of situations, including near a superconductor/normal boundary in a current-carrying wire where the divergence condition on $\vec{J}_s$ is not satisfied, or in a multiply-connected superconductor where the superconducting order parameter can no longer be taken to be purely real.